Problem Set 9 – Relativity (`GOI36A`)

17/11/2020

1 Einstein-Maxwell Theory

Einstein-Maxwell theory describes a massless, spin-1 particle (i.e. a *photon*) in a curved spacetime. For technical reasons we will not get into here, we cannot use a scalar field to describe this photon; instead, we have to use a vector field A_μ.

The action for the theory is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \ , \tag{1.1}$$

where R is the usual Ricci scalar and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. That is, the action has the usual Einstein-Hilbert term, as well as an additional matter action that involves derivatives of the photon field. Also, remember that the volume factor $d^4x \sqrt{-g}$ is necessary in order for the action to be invariant under coordinate transformations.

1.) Compute the stress-energy tensor for Einstein-Maxwell theory, and then write down the Einstein equation.

2.) Derive the equation of motion for the photon field.

3.) Prove the Bianchi identity $\nabla_{[\mu} F_{\nu\rho]} = 0$.

4.) Prove that the stress-energy tensor is conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$. You'll have to make use of your results from problems 2 and 3 to do this.

The Einstein equation, the photon equation of motion, and the Bianchi identity are all second-order partial differential equations that determine the dynamics of the metric $g_{\mu\nu}$ and the photon field A_μ. The presence of the photon field means that there are many new solutions allowed that would not be by the ordinary vacuum Einstein equations. In particular, since the photon mediates the electromagnetic force, Einstein-Maxwell theory allows for black holes that have electromagnetic charge! Such black holes are called *Reissner-Nordström black holes*, and we will study them in detail in upcoming classes.

Note: we've actually done problems 2, 3, and 4 in Minkowski spacetime on a previous problem set. On a general curved spacetime manifold, though, we have to keep track of covariant derivatives in order to generalize these proofs.

2 de Sitter from Minkowski

Consider a four-dimensional Minkowski spacetime:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ , \tag{2.1}$$

where we have set $c = 1$. In this problem, we will consider the hyperboloid surface in this spacetime given by

$$-t^2 + x^2 + y^2 + z^2 = \alpha^2 \ , \tag{2.2}$$

for some real, positive constant α. The three-dimensional spacetime spanned by this surface is Lorentzian and has constant positive curvature. Such a spacetime is known as a de Sitter spacetime. In particular, this is dS_3, i.e. a three-dimensional de Sitter spacetime.

Note that the hyperboloid condition (2.2) places one constraint on our four independent coordinates. This means that it is a three-dimensional subspace, and we will hence need three coordinates to parameterize it. In order to understand the induced metric on this hyperboloid, we first need to find a set of three coordinates that solves the hyperboloid condition (2.2).

5.) One set of coordinates we can use to parameterize dS_3 is called the *static patch*, with coordinates (u, ρ, ψ) defined by

$$\begin{aligned} t &= \sqrt{\alpha^2 - \rho^2} \sinh\left(\frac{u}{\alpha}\right) \ , \\ x &= \sqrt{\alpha^2 - \rho^2} \cosh\left(\frac{u}{\alpha}\right) \ , \\ y &= \rho \sin\psi \ , \\ z &= \rho \cos\psi \ . \end{aligned} \tag{2.3}$$

Prove that the static patch correctly satisfies the hyperboloid condition (2.2). Then, compute the induced metric on the static patch.

6.) Another way to approach to parameterize dS_3 is to first write Minkowski in spherical coordinates (t, r, θ, ϕ). Then, we can use the *closed slicing*, with coordinates (τ, θ, ϕ) defined such that

$$\begin{aligned} t &= \alpha \sinh\left(\frac{\tau}{\alpha}\right) \ , \\ r &= \alpha \cosh\left(\frac{\tau}{\alpha}\right) \ , \end{aligned} \tag{2.4}$$

while θ, ϕ still range over $(0, \pi)$ and $(0, 2\pi)$, respectively. Prove that the closed slicing describes the hyperboloid (2.2), and then compute the induced metric in this closed slicing.

7.) Consider the induced metric on the static patch. You should find that the metric blows up on the two-dimensional surface spanned by $\rho = \alpha$. Is this a true singularity, or is this just a coordinate singularity? Be as explicit as possible in proving your answer.

8.) **Bonus:** Write down a metric for dS_d, i.e. the d-dimensional de Sitter spacetime, by generalizing our construction of dS_3 to higher dimensions. (Hint: what is the natural generalization of the hyperboloid condition to higher dimensions?) Feel free to use any coordinate patch on de Sitter that you want.